

Strongly Interacting Atom Lasers in Three Dimensional Optical Lattices

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We show that the dynamical melting of a Mott insulator in a three-dimensional lattice leads to condensation at nonzero momenta, a phenomenon that can be used to generate strongly interacting atom lasers in optical lattices. For infinite onsite repulsion, the case considered here, the momenta at which bosons condense is determined analytically and found to have a simple dependence on the hopping amplitudes. The occupation of the condensates is shown to scale linearly with the total number of atoms in the initial Mott insulator. Our results are obtained using a Gutzwiller-type mean-field approach, gauged against exact diagonalization solutions of small systems.

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The invention of the first optical lasers more than half a century ago marked the beginning of the ultimate control over light waves in a way that would later revolutionize the world. Now, with the realization of Bose-Einstein condensation (BEC) in atomic gases [1], we are entering an era in which a similar degree of control over matter waves is being achieved. In recent years, much effort has been devoted to developing techniques for converting the trapped atoms of a BEC into freely propagating coherent matter waves – the so called atom lasers [2, 3]. The realization of atom lasers [3] has opened promising avenues of theoretical and experimental research. These coherent matter-wave beams are expected to become constituents in future scientific and technological devices [4].

So far, most of the atom-laser experiments have dealt with weakly-interacting gases. The possibility of gaining further control by enhancing interactions by means of optical lattices has remained open. Here we address this issue and demonstrate that the free expansion of bosons from a Mott-insulating state into an empty lattice leads to the emergence of coherent matter waves at nonzero momenta, which in turn can be fully controlled by the hopping amplitudes in the lattice. In one dimension (1D), a related phenomenon was reported in Ref. [5], where the melting of a Mott insulator of hard-core bosons (HCBs) was shown to produce a quasi-coherent matter wave characterized by power-law decaying correlations. Interestingly, the power law was found to be the same as for the ground-state solution, i.e., in this transient regime, the system cannot be characterized by a finite temperature. Analytical work in the XY chain reproduced this behavior [6], and further studies in 1D showed that soft-core bosons [7] and two-component fermionic systems [8] exhibit similar phenomena.

Following the promising results obtained for 1D systems, in this Letter we examine the dynamics of bosons in higher dimensions. For concreteness, we will focus on the hard-core limit [9]. Studying the expansion in higher dimensions is of much interest not only because most experiments are performed in these regimes but also because true condensation, and hence a true atom laser, can

only be realized in dimensions higher than one. We examine whether condensation (off-diagonal long-range order) can develop dynamically during the expansion of an initial state that exhibits only short-range correlations, namely, a Mott-insulating state.

In this study, we consider HCBs on a three dimensional (3D) lattice, with $N = L_x \times L_\perp^2$ sites in the presence of a general onsite potential. Here, L_x (L_\perp) denotes the number of sites in the \hat{x} (\hat{y} and \hat{z}) direction(s). The Hamiltonian can be written as:

$$\hat{H} = - \sum_{\langle ij \rangle} t_{ij} \left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i \right) - \sum_i \mu_i \hat{n}_i, \quad (1)$$

where $\langle ij \rangle$ denotes nearest neighbors, \hat{a}_i (\hat{a}_i^\dagger) destroys (creates) a HCB on site i , $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the local density operator, μ_i is the local chemical potential, and the t_{ij} 's are the hopping amplitudes. The HCB creation and annihilation operators satisfy the constraints $\hat{a}_i^{\dagger 2} = \hat{a}_i^2 = 0$, which preclude multiple site occupancies. Equation (1) may be viewed as a spin-1/2 XY model with a site-dependent magnetic field applied in the \hat{z} direction.

A typical initial state in our system is an almost perfect Mott insulator, created by placing HCBs in a very strong harmonic trap with curvature along the \hat{x} direction, i.e., with $\mu_i = \mu_0 - Vx_i^2$, where V is the curvature of the trap, x_i is the distance of the i -th site from the center of the trap along the \hat{x} direction, and μ_0 is the chemical potential at the center of the trap. The resulting initial state is a ‘slab’ of width W_x around the center of the trap in the \hat{x} direction with $\langle \hat{n}_i \rangle \approx 1$, which extends over the entire lattice in the transverse directions. In experiments, such a state could be produced by a box-like trap in the transverse directions and very strong onsite interactions ($U/t \gg 1$). For simplicity, we consider periodic boundaries in the transverse directions. This choice makes our system homogeneous in those directions as a result of which, the actual value of L_\perp ($L_\perp \geq 2$) plays no role in the mean-field analysis.

At time $\tau = 0$, the harmonic trap is turned off and the bosons start expanding freely throughout the lattice. Their particular initial state causes them to move

in the \hat{x} direction leaving the system translationally invariant in the transverse ones. We then study various properties of the system measured in the course of the evolution. Among those are density profiles, momentum distribution functions, and the condensate fraction along with the lowest natural orbital (LNO). The latter two are the largest eigenvalue of the one-particle density matrix $\rho_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle$ and its eigenvector, respectively [10].

Partial control over the dynamics of the system is gained by considering two different hopping parameters: The first is the hopping amplitude in the \hat{x} direction, denoted by t_x . Without loss of generality, this amplitude is fixed at $t_x = 1$, setting the energy scale. The second value corresponds to the hopping amplitude in the transverse directions $t_y = t_z$. For reasons that will become apparent later, we shall use the dimensionless quantity $\eta = (t_y + t_z)/t_x$ as the control parameter.

In contrast to the special 1D case, where the model has an exact solution [5], to explore the dynamics of HCBs in higher dimensions one must resort to approximate schemes. This is because numerically-exact methods such as exact diagonalization, scale exponentially with the system size and hence allow insight into the dynamics of only small systems. Here, we probe the dynamics of HCBs primarily by means of a Gutzwiller-type mean-field approximation, adjusted to handle time evolution via the time-dependent variational principle method, originally developed for soft-core bosons [11]. This approach becomes exact in the limit of infinite dimensions, but has proved to provide qualitatively correct phase diagrams for the ground state of two-dimensional (2D) and 3D hard-core boson systems [12].

Within this method, we employ the following simple product ansatz as the wave-function of the system:

$$|\psi\rangle = \prod_{j=1}^N e^{i\chi_j} \left(\sin \frac{\theta_j}{2} |0\rangle + \cos \frac{\theta_j}{2} e^{i\phi_j} |1\rangle \right), \quad (2)$$

where we have utilized the usual parametrization of spin-1/2 particles. The polar angles $\theta_i \in [0, \pi]$, and the azimuthal angles, χ_i and $\phi_i \in [0, 2\pi)$, are time-dependent. By stationarizing $\langle \psi | i\partial/\partial\tau - \hat{H} | \psi \rangle$ (here, $\hbar = 1$), we obtain the following set of first-order coupled equations:

$$\dot{\theta}_i = 2 \operatorname{Im}[S_i], \quad \dot{\phi}_i = \mu_i - 2 \operatorname{Re}[S_i] \cot \theta_i, \quad (3)$$

where $S_i \equiv \frac{1}{2} \sum_j t_{ij} \sin \theta_j e^{i(\phi_j - \phi_i)} \delta_{i,n(j)}$, and $\delta_{i,n(j)}$ is unity if i and j are neighbors and zero otherwise [13].

Equations (3) are then solved simultaneously for all sites using a multi-dimensional forth-order Runge-Kutta method with an $\mathcal{O}(\Delta\tau^5)$ error, where $\Delta\tau$ is the discrete time step (taken to be $\Delta\tau = 0.001$). Self consistency of the method is verified by monitoring the conservation of quantities such as total number of particles, total energy and total momentum. For the chosen time step, these quantities were observed to remain constant (up to 8 significant digits) throughout the entire simulation.

Since our description of the model is approximate in nature, in order to gain an understanding of the accuracy of our results, we compare them against exact solutions. The latter are available in 1D, where the mean-field approach is not expected to be accurate [14], and for small 2D systems where exact diagonalization calculations can be used.

In Fig. 1, we present an analysis of several small 2D systems. The figure shows the time evolution of the normalized difference for $\langle \hat{n}_{\mathbf{k}} \rangle$ between the mean-field results and the exact ones, defined as

$$e_{\text{mom}} = \frac{\sum_{\mathbf{k}} |\langle \hat{n}_{\mathbf{k}} \rangle_{\text{exact}} - \langle \hat{n}_{\mathbf{k}} \rangle_{\text{mean-field}}|}{\sum_{\mathbf{k}} \langle \hat{n}_{\mathbf{k}} \rangle_{\text{exact}}}, \quad (4)$$

for lattices with up to 6×4 sites and up to six bosons, and two different values of η [14].

As expected, the errors increase during the expansion and attain significant values in all cases (≈ 0.8 in the worst case and ≈ 0.15 in the best case) even before they start bouncing off the edges of the lattice (which occurs at $\tau \sim 1$). However, several conclusions may be drawn from comparison of the errors in the various cases. First, larger values of η yield smaller errors [Fig. 1(a) vs 1(b)]. This is not surprising considering that as η increases, the system moves from 1D ($\eta = 0$) to ‘full’ 2D ($\eta = 1$), and that mean-field theories are expected to become more accurate as the dimensionality of the system increases [12]. Figure 1 also shows that two other factors contribute to the reduction of the error, (i) adding particles to the system, and (ii) increasing the lattice size in the transverse direction [14].

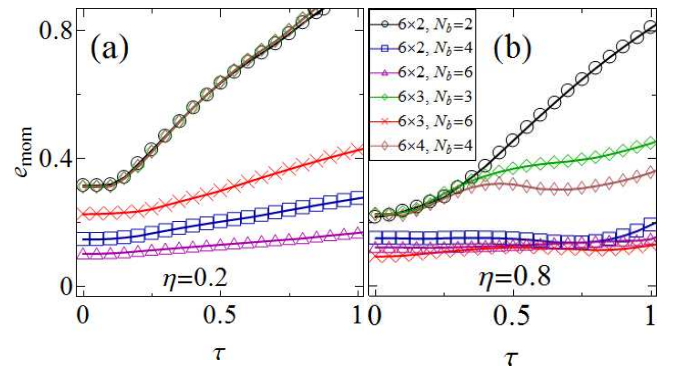


FIG. 1: (Color online) Momentum-distribution error e_{mom} [Eq.(4)] as a function of time τ for different small 2D systems of size $L_x \times L_y$ with N_b bosons. The hopping ratios are: (a) $\eta = 0.2$ and (b) $\eta = 0.8$ [14].

The fact that in 2D, the mean-field results are qualitatively similar to the ones obtained with exact diagonalization [14], and the quantitative decrease of the relative errors as η becomes large or as the number of particles increases, suggest that in 3D one can not only gain a qualitative understanding of the dynamics of the bosons during the melting of a Mott insulator, but also an actual

quantitative picture of it. This is both because mean-field is expected to be more accurate in 3D than in 2D, and because the systems we can simulate contain far more particles on much larger lattices.

Figure 2 depicts the main result of this Letter. It shows the density profiles and momentum distribution functions of the evolution of $N_b = 40 \times L_\perp^2$ bosons on a $700 \times L_\perp^2$ -site lattice, during the melting of a three-dimensional Mott insulator with $\eta = 0.8$. Several important features are apparent in these plots. In the density profiles (left panels), one can see that during the expansion, after the insulator has completely melted, two solitonic ‘lumps’ of bosons form and subsequently move in opposite directions with constant velocity. Even though the lumps slowly broaden as they travel, they retain their shape [compare Figs. 2(e) and 2(g)]. Cross-sectional plots of Fig. 2 are presented in the supplementary material.

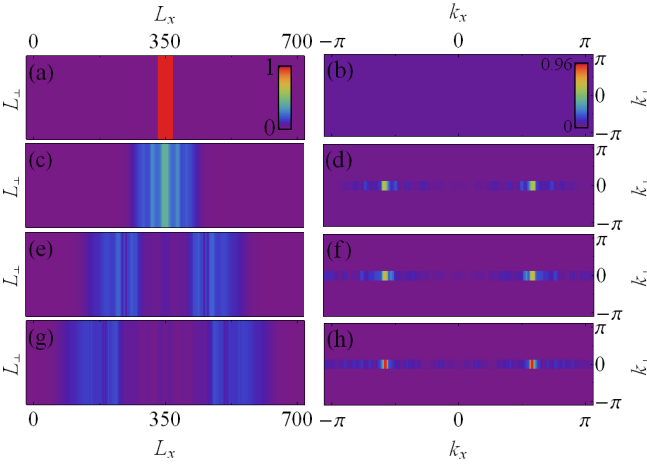


FIG. 2: (Color online) Evolution of: density profiles (left panels) and momentum distribution function, as a fraction of L_\perp^2 with the \hat{z} direction omitted (right panels), during the free expansion of $40 \times L_\perp^2$ HCBs in a 3D lattice with $L_x = 700$ and $\eta = 0.8$. The times shown here are, from top to bottom, $\tau = 0$, $\tau = 84$, $\tau = 151$ and $\tau = 187$.

The above behavior of the density profiles is accompanied by an even more remarkable behavior of the momentum distribution function (right panels in Fig. 2). While $\langle \hat{n}_{\mathbf{k}} \rangle$ of the initial state is almost completely flat (as the Mott insulator is localized in real space), as the system expands, interactions between bosons redistribute the energy. This results in the emergence of two condensates at opposite momenta. Given the dispersion relation in the lattice, these momenta determine the velocity of the two lumps seen in the density profiles [14].

As it turns out, the momenta at which condensation takes place can be evaluated analytically without resorting to mean-field theory. To calculate them, we first obtain the dispersion relation of the system in the absence of the trapping potential, given by

$$\epsilon_{\mathbf{k}} \equiv \langle \mathbf{k} | \hat{H} | \mathbf{k} \rangle = -2t_x \cos k_x - 2t_y \cos k_y - 2t_z \cos k_z \quad (5)$$

where $|\mathbf{k}\rangle = \hat{a}_{\mathbf{k}}^\dagger |0\rangle$ and $\hat{a}_{\mathbf{k}}^\dagger$ ($\hat{a}_{\mathbf{k}}$) creates (destroys) a HCB with momentum \mathbf{k} . Note that even though the dispersion relation is the same as for noninteracting particles, HCBs do interact due to the constraints on site occupancy. (This is reflected in Fig. 2, where momenta are redistributed during the expansion.) In our setup, the initial total energy of the system is $\epsilon \approx 0$, as can be immediately verified by considering the energy of the insulating ground state. Since the expansion takes place in the \hat{x} direction, it is expected that if condensation occurs it would take place at momenta $(\pm q_x, 0, 0)$, for some q_x which is to be determined. Further assuming that the majority of the bosons occupy these two modes, together with conservation of energy, leads [from Eq. (5)] to a simple relation between the momentum of the condensate (q_x) and the parameters of the model:

$$\cos q_x = -\eta. \quad (6)$$

This remarkable behavior of the HCB gas therefore implies that the locations of the two prominent momentum modes are fully controlled by the hopping parameters.

The mean-field results agree in full with this analytical argument. This is shown in Fig. 3(a) where we plot the locations of the momentum peaks for different values of η as predicted by Eq. (6) (solid line) and the result of the mean-field simulation (points). The velocity at which the condensates travel in the lattice is then given by $\mathbf{v}_g = \nabla \epsilon_{\mathbf{k}} = 2t_x \sin q_x \hat{x}$. Interestingly, if the hopping parameters are such that the right-hand-side of Eq. (6) is greater than unity, the bosonic gas does not melt within the mean-field theory. This is an artifact of the mean-field approach. Exact-diagonalization calculations on 2D systems show that the Mott insulator does slowly melt for $\eta > 1$. Hence, we expect that in large finite-dimensional systems with $\eta > 1$ the Mott insulator will melt but no condensation will occur. Since the 2D systems we can study with exact diagonalization are too small to ver-

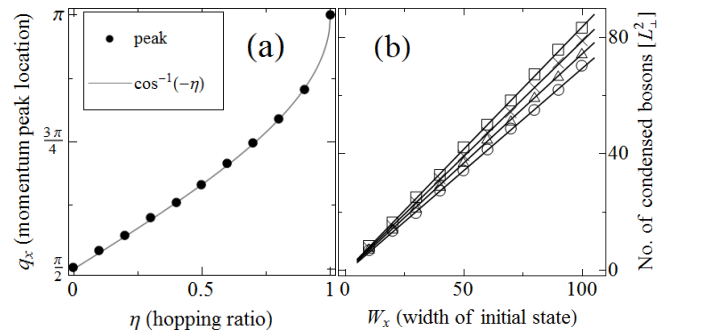


FIG. 3: (a) Dependence of the momentum peak locations on the hopping parameters. The data points are taken from the various simulations whereas the line is $\cos^{-1}(-\eta)$. (b) Scaling of the three-dimensional system: Total number of condensed bosons (in units of L_\perp^2) as a function of width of the initial Mott insulating state for $\eta = 0.2$ (\circ), $\eta = 0.4$ (\triangle), $\eta = 0.6$ (\times) and $\eta = 0.8$ (\square).

ify this hypothesis, this matter will have to be tested in experiments with ultracold bosons in optical lattices.

So far, we have been stating without proof that during the expansion of the initial Mott insulator, the emergence of peaks at nonzero momenta signals condensation. This was certainly not the case in the 1D systems studied in Ref. [5], even though similar peaks were observed at momenta $q_x = \pm\pi/2$. In 1D, the occupation of the two LNOs [which correspond to the two peaks in $n(k)$] was found to be proportional to $\sqrt{N_b}$, meaning that, in the thermodynamic limit, each condensate fraction (the ratio of the number of particles in each LNO to the total number of particles in the system) is zero [5]. True condensation only takes place if the LNO occupation is in proportion to the total number of particles in the system [15], hence one must verify that it scales with N_b .

In Fig. 3(b), we study the scaling of the total occupation of the LNOs in our systems for different values of η ranging from 0.2 to 0.8. As the figure indicates, the number of condensed bosons indeed grows linearly with the width W_x , i.e., with the total number of bosons in the initial Mott insulating state. As the value of η increases and one moves from a quasi-1D system to a more 3D system, the slope, that is, the fraction of bosons that condense, grows as well. While it is well known that true condensation can occur in 2D systems at zero temperature and in 3D or higher dimensional systems below some critical temperature, the surprising result here is that condensation is found out of equilibrium, in a system where the energy is conserved, and for an insulating initial state that exhibits no off-diagonal correlations.

The dynamics of the systems analyzed above describes bosons hopping on a 3D lattice with periodic boundary conditions in the transverse directions. These were chosen in order to eliminate edge effects and to keep the system homogeneous in those directions. This geometry is, unfortunately, not realizable experimentally. In experiments, the boundaries are naturally open (a box trap or other trapping potentials). In the supplementary material, we show that our conclusions are not altered when open boundaries (a box trap) are considered in the transverse directions provided that L_\perp is sufficiently large. This is to be expected since $L_\perp \rightarrow \infty$ and periodic boundary conditions in the \perp directions are equivalent. In practice, the results are found to be similar for $L_\perp \gtrsim 40$ [14], which are lattice sizes that are accessible experimentally.

In summary, we have shown that the melting of a Mott insulator with $\langle \hat{n}_i \rangle \approx 1$ in a three dimensional lattice leads to the dynamical emergence of condensates at nonzero momenta. We have determined the momenta at which condensation takes places and showed that these values have a simple dependence on the hopping amplitudes in the lattice. We have also shown that the occupation of these out-of-equilibrium condensates scales with the number of particles in the initial Mott insulator, and

with a proportionality constant that increases with η . We followed a mean-field approach that was gauged against exact diagonalization results of small 2D systems, where qualitatively similar results were obtained within both approaches.

Our results suggest that the expansion of a Mott insulator can be used to generate strongly interacting atom lasers in optical lattices, where the velocity of the condensates (and their wavelengths) can be fully controlled by the parameters of the optical lattice involved in the setup. This study thus presents evidence for a fundamentally new phenomenon, namely, out-of-equilibrium condensation in an isolated, and hence, energy-conserving system. As such, this study may have important implications on current experiments with ultracold Bose gas in optical lattices, as well as on our understanding of quantum systems out of equilibrium. We note here that experimental setups capable of verifying the results presented here are currently available [16]. As mentioned before, we have checked that this phenomenon also occurs in the soft-core boson case for large onsite repulsive interactions, in which case the dispersion relation, and hence the momenta at which the bosons condense, becomes dependent on the value of the onsite repulsion.

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Supplementary material for EPAPS Strongly Interacting Atom Lasers in Three Dimensional Optical Lattices

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MEAN-FIELD VS EXACT RESULTS IN 1D

In one dimension, the existence of an exact solution to our model of interest (see Ref. [5] in the Letter) enables us to compare mean-field results against exact ones in systems with lattices which are comparable in size with those realized experimentally. The 1D comparison should however be treated with caution because the mean-field approximation is expected to yield meaningful results

only in higher dimensions. Nonetheless, in Fig. 4 we show several snapshots of the density profile of a 1D system, taken at five different times. These show the mean-field local densities (triangles), compared against matching exact ones (circles). In both cases, 20 bosons are released from a trap with $V = 0.01t_x$ on a 200-site lattice.

It is clear from the figure that despite the visible quantitative differences between the mean-field evolution and the exact one, the two systems exhibit a qualitatively similar expansion. In both cases, two ‘lumps’ of bosons emerge during the melting of the Mott insulator, and subsequently move away from each other at the same constant velocities. Remarkably, the fronts of the exact and mean-field condensates propagate together, as can be verified by examining the points where the density goes to zero in both solutions.

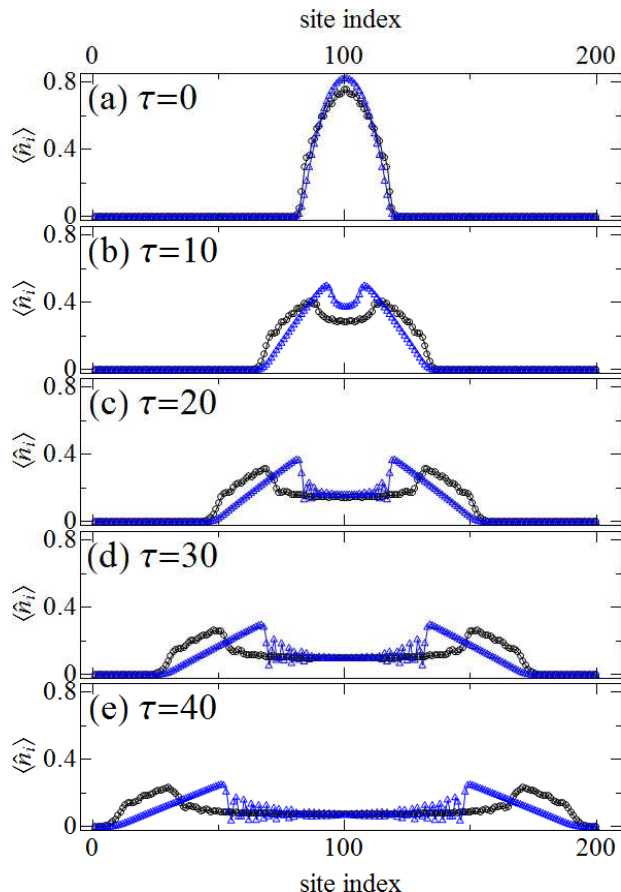


FIG. 4: Evolution of 20 HCBs on a 200-site 1D lattice. The HCBs are initially trapped inside a harmonic potential and are then released. The figure shows the density profiles of the system at five different times (from top to bottom: $\tau = 0, 10, 20, 30$ and 40) as it was computed by exact means (circles) and by the mean-field approximation (triangles).

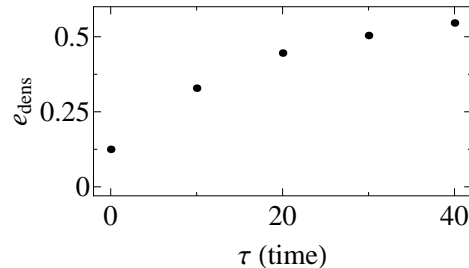


FIG. 5: The error e_{dens} [Eq. (7)] as a function of time for the mean-field evolution displayed in Fig. 4.

Figure 5 shows the normalized difference between the mean-field local densities and the exact ones, given by:

$$e_{\text{dens}} = \frac{\sum_i |\langle \hat{n}_i \rangle_{\text{exact}} - \langle \hat{n}_i \rangle_{\text{mean-field}}|}{\sum_i \langle \hat{n}_i \rangle_{\text{exact}}}. \quad (7)$$

The errors clearly increase with time. However, they do not increase as fast as may have naively been expected for a mean-field solution applied to a 1D system. This is encouraging in terms of the relevance of the mean-field approach for this model when applied to higher dimensional systems.

MEAN-FIELD VS EXACT RESULTS IN 2D

Here, we discuss in further detail the exact setup through which the analysis of the momentum-error presented in the Letter [see Eq. (4) there] was performed. We also extend the analysis by considering the behavior of the error for two additional values of the parameter η and discuss a similar analysis performed on the errors of the local densities [Eq. (7)]. This is done in order to strengthen the conclusions drawn in the text in that context.

Since in 2D, exact diagonalization methods allow insight into the dynamics of systems with only a small number of particles hopping on small-size lattices, the setup for the error analysis presented here is slightly different than the basic scenario discussed in the Letter where bosons were released from a trap whose origin was the center of a periodic lattice. Here, we have used a harmonic potential along the \hat{x} -direction centered around one edge of the lattice, supplemented by open boundary conditions in the \hat{x} direction. This setup takes advantage of the parity symmetry ($x \rightarrow -x$) of the expansion observed in the conventional setup. This method has been used in the time-dependent DMRG calculations in Ref. [8] in the Letter. In order to make the behavior of the different systems comparable, the characteristic density (see Ref. [5] in the Letter) of the different initial states was kept fixed for all the systems examined. This was done by choosing the curvature of the harmonic potential to be $V = V_0/W_x^2$ (in the figures, $V_0 = 2$, and $W_x = 1, 2$, or 3). In Fig. 1 in the Letter, and in Figs. 6 and 7 here, the behavior of the error is shown up to $\tau = 1$, which is the typical time it takes for the particles to reach the opposite edge of the lattice and bounce back.

In the Letter, we have shown that, in general, as the value of η increases, the discrepancies between the mean-field and exact momentum profiles decrease. There, this was shown for two values of η , namely, $\eta = 0.2$ and $\eta = 0.8$. To complete this picture and to provide further corroboration of our conclusions, in Fig. 6 we display the behavior of the momentum-error as a function of time for the same small 2D lattices but with $\eta = 0.4$ and $\eta = 0.6$.

In accord with the conclusions reported in the text, it can be seen in Fig. 6 that larger values of η yield smaller errors [Fig. 6(a) vs 6(b)], but also that adding particles and/or sites in the transverse direction both reduce the errors as well. The two figures fit with the trend set by the $\eta = 0.2$ and $\eta = 0.8$ panels of Fig. 1 in the Letter, and therefore strengthen our previous argument that for many-particle 3D systems the mean-field approach may be a good approximation for the study of the dynamics of 3D bosons.

Further support to our conclusions is provided by the behavior of the errors of the local densities. These are computed via Eq. (7), and depicted in Fig. 7 for the four

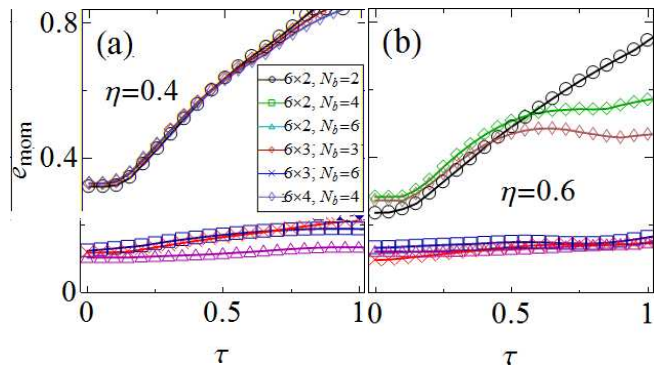


FIG. 6: (Color online) Momentum-error e_{mom} [Eq. (4) in the Letter] as a function of time τ for different small 2D systems of size $L_x \times L_\perp$ with N_b bosons. The hopping ratios are: (a) $\eta = 0.4$ and (b) $\eta = 0.6$. As the figures indicate, the errors in the $\eta = 0.6$ case are smaller, since larger values correspond to systems which are more two-dimensional. Also, as the number of particles in the system or the number of lattice sites in the transverse direction increase, the errors become smaller. These results are in accord with those shown in Fig. 1 in the Letter.

different values of η . Here too we have found that the errors decrease with increasing η , and that increasing the lattice size in the transverse direction and the addition of particles both reduce the difference between the mean-field and exact results.

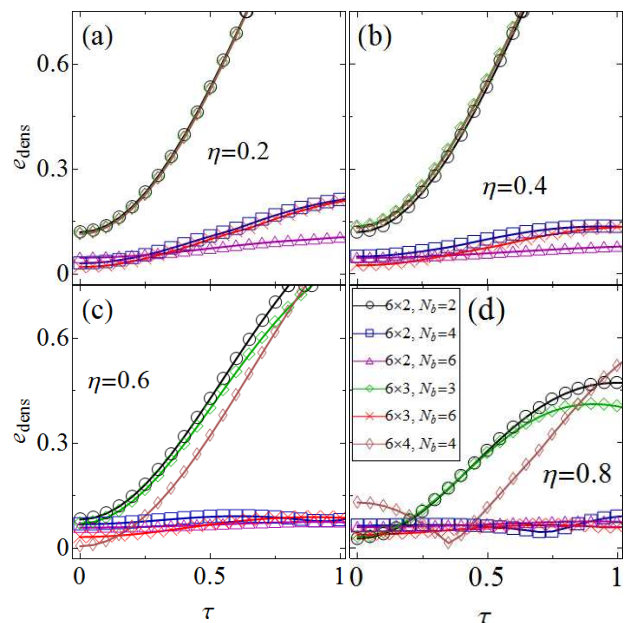


FIG. 7: (Color online) The error e_{dens} [Eq. (7)] as a function of time τ for different small 2D systems of size $L_x \times L_\perp$ with N_b bosons. The hopping ratios are: (a) $\eta = 0.2$, (b) $\eta = 0.4$, (c) $\eta = 0.6$, and (d) $\eta = 0.8$.

CROSS SECTIONS

In Fig. 2 of the Letter, we presented intensity plots of the density and momentum profiles of the bosonic gas during the expansion of a Mott insulator in 3D. In Fig. 8, we provide the cross sections of the same density and momentum profiles, with cuts along the x and k_x axes, i.e., $y = z = 0$ and $k_y = k_z = 0$, respectively.

As evident from Fig. 8(a), when the initially-trapped bosonic gas is released, two solitonic lumps begin to form during the melting of the Mott insulator. Once the latter has disappeared, the lumps start moving away without a considerable change in their shape while they slowly broaden. The cuts across the momentum distribution function [Fig. 8(b)] show very sharp peaks emerging at finite momenta. Those peaks signal condensation, as discussed in the text, where the lowest natural orbital occupation was found to scale proportionally with the number of particles in the system.

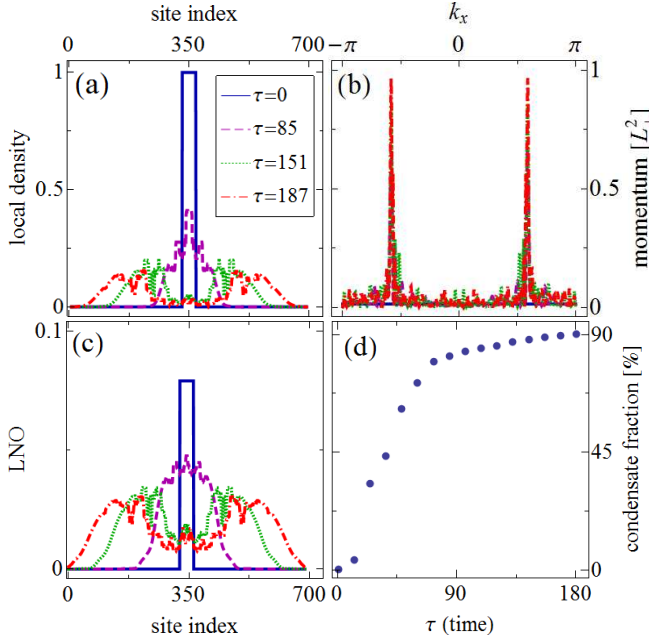


FIG. 8: (Color online) Evolution of (a) density profiles (b) momentum profiles (c) absolute value of the LNO wave function during the free expansion of a Mott insulator with $W_x = 40$ in a 3D lattice with $L_x = 700$ and $\eta = 0.8$. The times shown here are $\tau = 0$, $\tau = 84$, $\tau = 151$ and $\tau = 187$. (d) Total number of condensed bosons (LNO occupation) in the system divided by the total number of bosons, as a function of time.

In Fig. 8(c), we show the cross section along the x axis of the lowest natural orbital. The figure illustrates that the condensates are mainly localized in, and have similar shapes as, the two lobes visible in the density profiles. This explains why the (constant) group velocity of the condensates, determined by their momenta and the dispersion relation in the lattice (see related discussion in

the Letter), is equal to the velocity of motion of the lobes appearing in the density profiles.

The condensate fraction, i.e., the ratio between the occupation of the LNO and the total number of bosons, is shown in Fig. 5(d) as a function of time. There, one can see that it quickly increases in the initial stages of the expansion, during the melting of the Mott insulator, and then, after the insulator has melted, it exhibits a slow increase during the remainder of the evolution. The number of condensed bosons at the kink connecting the two regimes is the one reported in Fig. 3(b) of the Letter. Interestingly, while the expansion of the Mott insulator in our selected geometry can only cause the emergence of two independent condensates traveling in opposite directions (during their formation, the insulating slab between them precludes any phase correlations), the diagonalization of the one-particle density matrix computed within the mean-field approximation yields only one large eigenvalue, i.e., mean-field merges the two LNOs into one. Hence, the mean-field LNO occupation reflects the total number of condensed bosons (that is, the sum of the occupations of the two moving condensates). This can be easily verified by examining the evolution of only one half of the system described in the previous section.

EFFECTS OF OPEN BOUNDARY CONDITIONS

In order to verify that open boundary conditions in the transverse direction do not alter the general features of the systems studied in the Letter, we have tested the effects of open boundaries on the condensate occupation in the lattice as a function of the transverse linear system size L_\perp . This was done in order to illustrate that in large systems edge effects do not destroy the general phenomenon described earlier. The results of this investigation are summarized in Fig. 9. The data points represent the condensate fraction at a fixed time ($\tau = 18$) after the trap has already been removed and the bosons were allowed to expand and condense.

We have tested the effects of the open boundaries on the condensation in several 2D systems all with 100 sites in the \hat{x} direction but with varying L_\perp (here, $\eta = 0.2$). The dashed horizontal line marks the same quantity in the periodic ($L_\perp \rightarrow \infty$ limit) case discussed in the Letter. As expected, the larger L_\perp is, the less pronounced edge effects become and we can therefore conclude that provided L_\perp is large enough, systems with open boundary conditions (a box trap) in the transverse direction will exhibit the exact same features of parallel periodic systems. This will remain true in the time scales relevant for the expansion of the bosons or the formation of the solitonic condensates. The inset in the figure is a log-log scale of the data points, illustrating that the condensate occupation approaches the periodic value algebraically, where the exponent here is ≈ -0.675 .

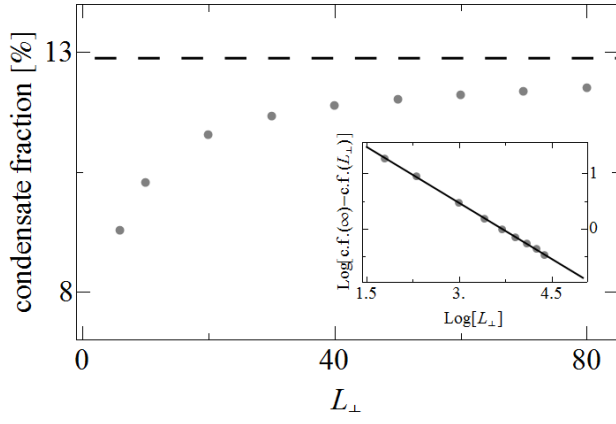


FIG. 9: Effects of open boundary conditions. Condensate fraction (given in %) at $\tau = 18$ as a function of lattice size in the transverse direction, for a 2D system with $L_x = 100$ and $\eta = 0.2$. As the size of the system in the transverse directions increases, the condensate fraction approaches the infinite-size (periodic) value marked by the dashed horizontal line. The inset shows a linear fit on a log-log scale illustrating the power-law behavior of the condensate fraction as it reaches the periodic limit. The exponent here is ≈ -0.675 .